— Reading —

1. Read text from p.418 (after proof of Cor. 9.3.2) to 422 (Th. 9.3.3)

— Exercise —

2. **Reparametrization by arc length.** Let I be an interval of $\mathbb{R}$ and $u : I \to \mathbb{R}^n$ a regular $\mathcal{C}^1$ application (i.e. a $\mathcal{C}^1$ application such that $u'(t) \neq 0$ for every t). We fix a point $t_0 \in I$ and define the arc length s by $s(t) = \int_{t_0}^{t} ||u'(x)||dx$. We define the curve Γ as $\Gamma = u(I)$.

   (a) Show that s is a $\mathcal{C}^1$ function on I and that $s'(t) = ||u'(t)||$ for every t.

   (b) Show that s is a bijective function onto $J := s(I)$ and that s^{-1} is $\mathcal{C}^1$ on J.

   (c) Finally, we define v on J by $v(t) = u(s^{-1}(t))$. Check that v is $\mathcal{C}^1$ on J, that $v(J) = \Gamma$ and that $||v'(t)|| = 1$ for every $t \in J$. *Hint: Use the relation $v \circ s = u$.*

— Problems —

3. **Holomorphic function.** Let U be an open set in $\mathbb{R}^2$ and $f = (f_1, f_2) : U \to \mathbb{R}^2$ a differentiable application. Let (a, b) be an element of U.

   (a) Recall the form of the matrix of a rotation with center 0 in $\mathbb{R}^2$ with respect to an orthonormal basis.

   (b) We assume that $Df(a, b)$ is a direct similarity with center $(0, 0)$ (i.e. the composition of a rotation and a homothety, both with center $(0, 0)$). Show that $\frac{\partial f_1}{\partial x}(a, b) = \frac{\partial f_2}{\partial y}(a, b)$ and $\frac{\partial f_1}{\partial y}(a, b) = -\frac{\partial f_2}{\partial x}(a, b)$.

   (c) We assume that f is $\mathcal{C}^2$ and that $Df(a, b)$ is a direct similarity with center $(0, 0)$. Show that the Laplacian of f_1 and f_2 at (a, b) is zero.

   (d) We can associate to f a complex-valued function F as follow: for $z = x + iy$, we put $F(z) = f_1(x, y) + if_2(x, y)$. Show that f is differentiable at (a, b) and that $df(a, b)$ is a direct similarity with center $(0, 0)$ if and only if the ratio $\frac{F(z)-F(a+ib)}{z-(a+ib)}$ has a finite limit when z tends to $a + ib$. In this case, we note $F'(a + ib)$ this limit. Express the angle and the ratio of $Df(a, b)$ in function of $F'(a + ib)$.

4. **Differential calculus and statistics.** One of the main tasks in statistics is to make an estimation of a parameter θ for a population $\left\{ y_i \mid i \in \mathbb{Z}_{>0} \right\}$ based on an observation of sample $\{y_1, \ldots, y_n\}$ of the population. An *estimator* for the actual value θ_0 is a "function" $\widehat{\theta}(y_1, \ldots, y_n)$ which converges to θ_0 when n tends to ∞. The likehood of the system is the product $L(y_1, \ldots, y_n, \theta) = \prod_{i=1}^{n} l(y_i, \theta)$, where l is the density of the law describing the distribution of θ in the population. A function $\widehat{\theta}(y_1, \ldots, y_n)$ which maximizes the number $L(y, \widehat{\theta}(y))$ for every $y = (y_1, \ldots, y_n)$ is considered to be a good estimator and is called a *maximum likehood estimator*.

   (a) We suppose that $l(y, \theta) = \frac{1}{\sqrt{\pi}} exp(-(y - \theta)^2)$. Show that, for every $(y_1, \ldots, y_n)$ fixed, the likehood $\theta \mapsto L(y_1, \ldots, y_n; \theta)$ admits a global maximum obtained for $\theta = \frac{y_1 + \cdots + y_n}{n}$.

   (b) Find a maximum likehood estimator for $l(y, \theta) = \theta exp(-\theta y)$ when $y > 0$ and $l(y, \theta) = 0$ otherwise.